Announcements

1) Last exam Thursday after Thanksgiving

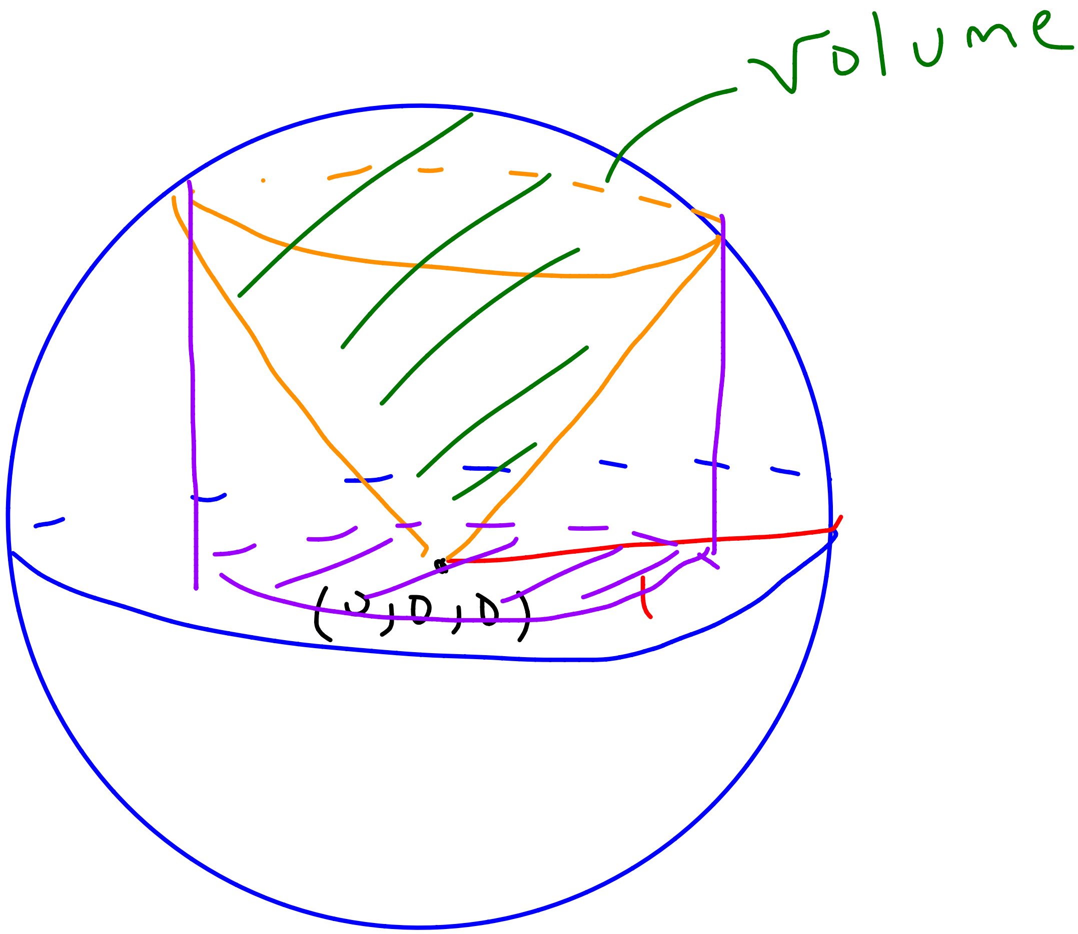
Example 1: (#25, 15.4)

Find the volume above

$z = \sqrt{x^2+y^2}$ and below

$x^2+y^2+z^2=1$

Picture



purple circle = region of integration

Inside Purple Circle

In polar coordinates,

$$0 \leq \theta < 2\pi$$

$$0 \leq r < \boxed{?}$$

To find upper bound, write the equation of the cone in polar coordinates.

$x = r\cos\theta \ , \ y = r\sin\theta$

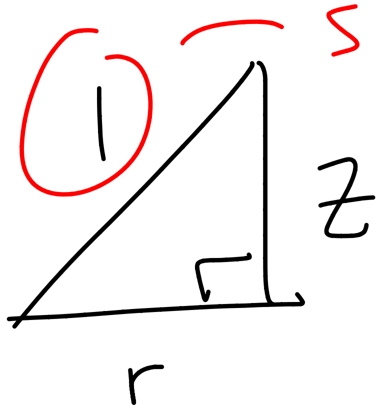
$$z = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \sqrt{r^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}}$$

$$= \sqrt{r^2} = r \quad (\text{assuming } r \geq 0)$$

$z = r$, always get a

45-45-90 triangle



since radius of sphere is one

$z = r$, so by Pythagorean Theorem,

$$2r^2 = 1$$

$$r^2 = \frac{1}{2}$$

$$r = \frac{1}{\sqrt{2}}$$

Region of Integration:

$$\boxed{\begin{aligned} 0 \le \theta < 2\pi \\ 0 < r \le \frac{1}{\sqrt{2}} \end{aligned}}$$

Compute

$$\int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} (\text{sphere} - \text{cone})\, r \, dr\, d\theta$$

do not forget!

Equation of sphere in polar coordinates:

$$z^2 + x^2 + y^2 = 1$$

$$z^2 + r^2\cos^2\theta + r^2\sin^2\theta = 1$$

$$z^2 + r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) = 1$$

$$z^2 + r^2 = 1$$

$$z = \pm\sqrt{1 - r^2}$$

Only top part of sphere, so

$$\boxed{z = \sqrt{1 - r^2}}$$

Volume:

$$\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\left(\sqrt{1-r^2}-r\right)r\,dr\,d\theta$$

$$=\int_0^{1/\sqrt{2}}\left(r\sqrt{1-r^2}-r^2\right)dr\cdot\int_0^{2\pi}1\,d\theta$$

$$=2\pi\left(\int_0^{1/\sqrt{2}}r\sqrt{1-r^2}\,dr-\int_0^{1/\sqrt{2}}r^2\,dr\right)$$

$$=2\pi\left(\underbrace{-\frac{(1-r^2)^{3/2}}{3}}_{\text{substitution, } u=1-r^2}-\frac{r^3}{3}\right)\Bigg|_0^{1/\sqrt{2}}$$

$$2\pi\left(-\frac{(1-r^2)^{3/2}}{3}-\frac{r^3}{3}\right)\Bigg|_0^{1/\sqrt{2}}$$

$$=2\pi\left(\left(-\frac{1}{6\sqrt{2}}-\frac{1}{6\sqrt{2}}\right)+\frac{1}{3}\right)$$

$$=\boxed{2\pi\left(\frac{1}{3}-\frac{1}{3\sqrt{2}}\right)}>0 \checkmark$$

Example 2: (#53, 15.3)

$$\int_0^1 \int_{\arcsin(y)}^{\pi/2} \cos(x)\sqrt{1+\cos^2(x)}\, dx\, dy$$

Change bounds of integration.

Draw the region

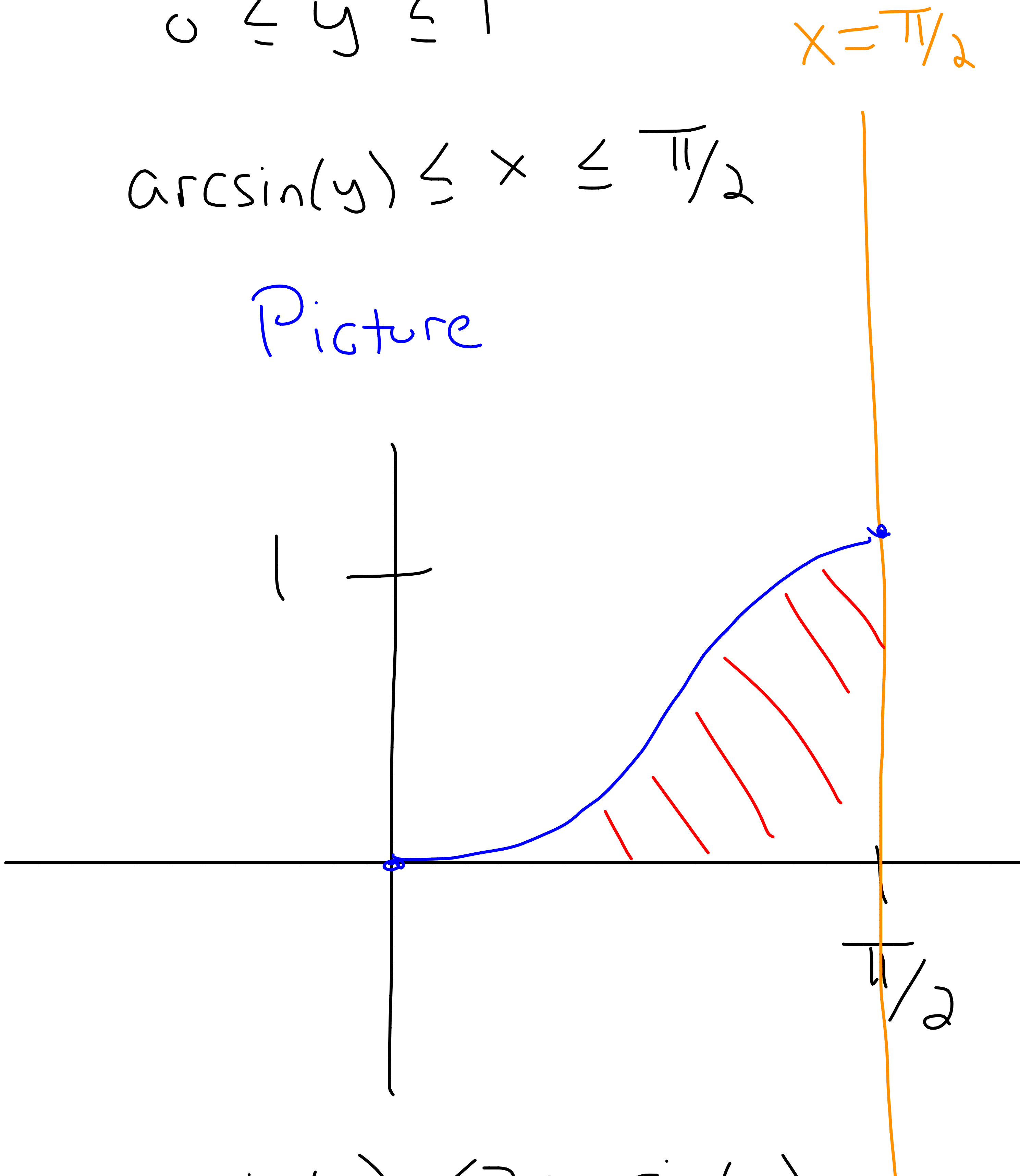
$$0 \le y \le 1$$

$$\arcsin(y) \le x \le \pi/2$$

$0 \le y \le 1$

$\arcsin(y) \le x \le \pi/2$

Picture



$x = \arcsin(y) \Leftrightarrow y = \sin(x)$

We can also describe the region as

$$0 \le x \le \pi/2$$

$$0 \le y \le \sin(x)$$

Integrate!

$$\int_0^{\pi/2} \int_0^{\sin(x)} \cos(x)\sqrt{1+\cos^2(x)}\, dy\, dx$$

$$= \int_0^{\pi/2} \cos(x)\sqrt{1+\cos^2(x)} \left(\int_0^{\sin(x)} 1\, dy \right) dx$$

$$= \int_0^{\pi/2} \cos(x)\sqrt{1+\cos^2(x)}\left(\int_0^{\sin(x)} 1\,dy\right)dx$$

$$= \int_0^{\pi/2} \sin(x)\cos(x)\sqrt{1+\cos^2(x)}\,dx$$

$$u = 1+\cos^2(x)$$

$$du = -2\cos(x)\sin(x)\,dx$$

New bounds $u=2$ to $u=1$

$$= -\frac{1}{2}\int_{2}^{1} \sqrt{u}\, du$$

$$= \frac{1}{2}\int_{1}^{2} \sqrt{u}\, du$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}u^{3/2}}{3} \Big|_{1}^{2}$$

$$= \boxed{\frac{2\sqrt{2}}{3} - \frac{1}{3}}$$

Transformations of Euclidean Space

(section 15.10)

Going some way to explain the "r" in polar coordinate change of variables

Example 3: Define

$$T : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$T(x,y) = (2x, 2y)$$

Let $R = [0,1] \times [0,1]$,

$$f(x,y) = 3.$$

Compare:

$$\int_R f(2x, 2y)\,dA$$

vs.

$$\int_{T(R)} f(x, y)\,dA$$

$$T(R) = [0,2] \times [0,2]$$

$$\iint_R f(2x, 2y)\, dA$$

$$= \int_0^1 \int_0^1 3 \, dx\, dy$$

$$= 3 \int_0^1 1\, dx \cdot \int_0^1 1\, dy$$

$$= \boxed{3}$$

$$\int_{T(R)} f(x,y)\,dA$$

$$= \int_0^2 \int_0^2 3\,dx\,dy$$

$$= 3 \int_0^2 1\,dx \int_0^2 1\,dy$$

$$= \boxed{12} \neq 3 \,!$$

The integrals don't match,
we have a problem!

Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T: \mathbb{R}^n \to \mathbb{R}^m$. We can write T as

$$T((x_1, x_2, x_3, \ldots, x_n))$$

$$= (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

where

$$f_i : \mathbb{R}^n \to \mathbb{R}$$

for all $1 \le i \le m$.

So for $T : \mathbb{R}^2 \to \mathbb{R}^2$,

$$T(x,y) = (2x, 2y),$$

$$f_1(x,y) = 2x$$

$$f_2(x,y) = 2y$$

Example 4: Again, let

$$T(x,y) = (2x, 2y).$$

Let $R = [0,1] \times [0, 2\pi]$

Show

$$\iint_{T(R)} \sin(2x+y)\,dA$$

$$= 4 \iint_{R} \sin(4x+2y)\,dA$$

$$\iint_R \sin(4x+2y)\,dA$$

$$= \int_0^1 \left(\int_0^{2\pi} \sin(4x+2y)\,dy \right) dx$$

$$= \int_0^1 \left(-\frac{\cos(4x+2y)}{2} \Big|_0^{2\pi} \right) dx$$

$$= \frac{1}{2} \int_0^1 \left(\cos(4x) - \cos(4x+4\pi) \right) dx$$

$$= \frac{1}{2} \left(\frac{\sin(4x)}{4} - \frac{\sin(4x+4\pi)}{4} \right) \Big|_0^1$$

$$\frac{1}{2}\left(\frac{\sin(4x)}{4}-\frac{\sin(4x+4\pi)}{4}\right)\Big|_0^1$$

$$=\boxed{\frac{1}{8}\left(\sin(4)-\sin(4+4\pi)\right)}$$

$$T(R) = [0,2] \times [0,4\pi]$$

$$\int_0^2 \left(\int_0^{4\pi} \sin(2x+y)\,dy \right) dx$$

$$= \int_0^2 \left(-\cos(2x+y) \Big|_0^{4\pi} \right) dx$$

$$= \int_0^2 \left(\cos(2x) - \cos(2x+4\pi) \right) dx$$

$$= \left(\frac{\sin(2x)}{2} - \frac{\sin(2x+4\pi)}{2} \right) \Big|_0^2$$

$$= \left(\frac{\sin(2x)}{2} - \frac{\sin(2x+4\pi)}{2} \right) \Big|_0^2$$

$$= \boxed{\frac{1}{2}\left(\sin(4) - \sin(4+4\pi)\right)}$$

$$= \frac{1}{4} \int_R \sin(4x+2y)\, dA$$

In general

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (g(x,y), h(x,y))$

Let

$$J_T(x,y) = \det\left(\begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} \end{bmatrix}\right)$$

Let R be a region in $\mathbb{R}^2$,

$f, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ continuous on R.

If T is $\boxed{\text{one-to-one}}$ on R and $J_T \neq 0$ on R,

$$\boxed{\iint_{T(R)} f(x,y)\,dA = \iint_R f(T(x,y))\,|J_T(x,y)|\,dA}$$